How to Speed up the Sinc Series

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Though the sinc series is the key for processing signals in the discrete domain, its performance as interpolator is poor due to the slow decrease rate of the sinc tails. The solution to this problem has been available for decades but is fairly unknown in the signal processing field.

The solution in the sequel is an implementation of the method in

> J. J. Knab, "Interpolation of band-limited functions using the approximate prolate series," IEEE Trans. Inf. Theory, vol. IT-25, pp. 717-720, Nov. 1979.

Given a signal s(t) with two-sided bandwidth B, the Sampling Theorem provides the reconstruction formula

$$s(nT+u)=\sum_{p=-\infty}^{\infty} s((n-p)T)\mathrm{sinc}(u/T+p), \tag{1}$$

where T is the sampling period, n an integer, and u the shift relative to the sampling grid of t, i.e, $t=nT+u$ with $-T/2\leq u<T/2$. However, if this series is truncated at indices $\pm P$,

$$s(nT+u)\approx\sum_{p=-P}^{P} s((n-p)T)\mathrm{sinc}(u/T+p), \tag{2}$$

the accuracy is poor due to the tails of the neglected sinc pulses. To improve accuracy, we can make good use of the sampling inefficiency as follows.

Consider a pulse $w(t)$ whose main time lobe in inside the range $]-PT-T/2, PT+T/2[$, i.e, $w(t)\approx 0$ if $|t|\geq PT+T/2$, where P is the index at which we plan to truncate the series. For a small delay d, $|d|\leq T/2$, this condition implies that $w(t-d)\approx 0$ whenever $|t|\geq (P+1)T$. Additionally, if $w(t)$ has two-sided bandwidth B_w and $(B+B_w)T<1$ (Nyquist condition), then (1) can be applied to the product $s(nT+u)w(u-d)$,

$$s(nT+u)w(u-d)=\sum_{p=-\infty}^{\infty} s((n-p)T)w(-pT-d)\mathrm{sinc}(u/T+p). \tag{3}$$

and the truncated formula is

$$s(nT+u)w(u-d)\approx\sum_{p=-P}^{P} s((n-p)T)w(-pT-d)\mathrm{sinc}(u/T+p). \tag{4}$$

In contrast with (2), this formula is accurate since the tails of the neglected terms have been damped by the samples $w(-pT-d)$, $|p|>P$. Finally, if $w(0)=1$ and we set $d=u$, we get

$$s(nT+u) \approx \sum_{p=-P}^{P} s((n-p)T)w(-pT-u)\mathrm{sinc}(u/T+p). \tag{5}$$

This is the accurate interpolator that we were looking for. To simplify notation, define the pulse

$$g(t) = w(-t)\mathrm{sinc}(t/T). \tag{6}$$

Then (5) reads

$$s(nT+u) \approx \sum_{p=-P}^{P} s((n-p)T)g(pT+u). \tag{7}$$

This formula says that we must replace the sinc pulse by $g(t)$ to obtain accuracy. As to the pulse $w(t)$, the one used in the previous reference is the inverse Fourier transform of the Kaiser-Bessel window,

$$w(t) = \frac{\mathrm{sinc}\Big((1-BT)\sqrt{(t/T)^2-P^2}\Big)}{\mathrm{sinc}\big(j(1-BT)P\big)}. \tag{8}$$

This window has an excellent performance. Actually, while the error for the sinc pulse decreases as $1/P$, that of $g(t)$ decreases as $e^{-j\pi P(1-BT)}$!!. So, for the latter, small P values give very high accuracy. This behavior is due to the fact that $w(t)$ is an approximation of the first prolate wave function, which is the band-limited signal with the highest energy concentration in a finite interval.

The following code tests the accuracy of $g(t)$ versus that of $\mathrm{sinc}(t/T)$ for several P.