Data-flipping

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Let x_n be a real sequence of length N with Fourier transform $X(\omega) = U(\omega) + jV(\omega)$, where $U(\omega)$ and $V(\omega)$ are the real and imaginary parts of the Fourier transform, respectively. The magnitude squared of the Fourier transform of x_n is then given by $|X(\omega)|^2 = U(\omega)^2 + V(\omega)^2$.

Another real sequence y_n of length $2N$ is constructed from x_n by appending a flipped version of x_n to x_n. That is,

$$y_n = \begin{cases} x_n & , \quad \text{for} \quad n = 0...N-1 \\ x_{2N-n-1} & , \quad \text{for} \quad n = N...2N-1 \end{cases} \tag{1}$$

The Fourier transform of y_n is

$$\begin{aligned}
Y(\omega) &= \sum_{n=0}^{2N-1} y_n e^{-j\omega n} && (2)\\
&= \sum_{n=0}^{N-1} x_n e^{-j\omega n} + \sum_{n=N}^{2N-1} x_{2N-n-1} e^{-j\omega n} && (3)\\
&= X(\omega) + \sum_{n=0}^{N-1} x_{N-n-1} e^{-j\omega(n+N)} && (4)\\
&= X(\omega) + e^{-j\omega N} \sum_{n=0}^{N-1} x_{-(n-(N-1))} e^{-j\omega n} && (5)\\
&= X(\omega) + e^{-j\omega N} e^{-j\omega(N-1)} X(-\omega) && (6)\\
&= X(\omega) + e^{-j\omega(2N-1)} X(-\omega) && (7)
\end{aligned}$$

where the time shift and time reversal properties of the Fourier transform has been used. Because x_n is a real sequence, we have $X(-\omega) = U(\omega) - jV(\omega)$. We can then get an expression for $Y(\omega)$ in polar form

$$\begin{aligned}
Y(\omega) &= U(\omega) + jV(\omega) + e^{-j\omega(2N-1)} \left(U(\omega) - jV(\omega)\right) && (8)\\
&= U(\omega)\left(1 + e^{-j\omega(2N-1)}\right) + jV(\omega)\left(1 - e^{-j\omega(2N-1)}\right) && (9)\\
&= e^{-j\omega\frac{(2N-1)}{2}} \left(U(\omega)\left(e^{j\omega\frac{(2N-1)}{2}} + e^{-j\omega\frac{(2N-1)}{2}}\right) + jV(\omega)\left(e^{j\omega\frac{(2N-1)}{2}} - e^{-j\omega\frac{(2N-1)}{2}}\right)\right) && (10)\\
&= 2e^{-j\omega\frac{(2N-1)}{2}} \left(U(\omega)\cos\left(\omega\frac{(2N-1)}{2}\right) - V(\omega)\sin\left(\omega\frac{(2N-1)}{2}\right)\right) && (11)
\end{aligned}$$

where from the magnitude spectrum of $Y(\omega)$ is directly available. We can then write for the power spectrum of y_n

$$
\begin{aligned}
|Y(\omega)|^2 &= 4(U(\omega)^2+V(\omega)^2)\frac{\left(U(\omega)\cos\left(\omega\frac{(2N-1)}{2}\right)-V(\omega)\sin\left(\omega\frac{(2N-1)}{2}\right)\right)^2}{U(\omega)^2+V(\omega)^2} && (12)\\
&= 4|X(\omega)|^2K(\omega) && (13)
\end{aligned}
$$

For the function $K(\omega)$ we have

$$
\begin{aligned}
K(\omega) &= \frac{U^2\cos^2(\omega Q)+V^2\sin^2(\omega Q)-2UV\cos(\omega Q)\sin(\omega Q)}{U^2+V^2} && (14)\\
&= \frac{U^2(1-\sin^2(\omega Q))+V^2(1-\cos^2(\omega Q))-2UV\cos(\omega Q)\sin(\omega Q)}{U^2+V^2} && (15)\\
&= \frac{U^2+V^2-\left(U^2\sin^2(\omega Q)+V^2\cos^2(\omega Q)+2UV\cos(\omega Q)\sin(\omega Q)\right)}{U^2+V^2} && (16)\\
&= \frac{U^2+V^2-\left(U\sin(\omega Q)+V\cos(\omega Q)\right)^2}{U^2+V^2} && (17)\\
&\leq 1 && (18)
\end{aligned}
$$

So the function $K(\omega)$ that is multiplied to the power spectrum of x_n oscillates between 0 and 1. In fact, $K(\omega)$ can be written

$$
K(\omega)=\sin^2\left(\omega\frac{(2N-1)}{2}+\theta(\omega)\right) \qquad (19)
$$

where

$$
\theta(\omega)=\begin{cases}\arcsin\left(\frac{X}{\sqrt{X^2+Y^2}}\right) & ,\quad \text{for} \quad X\geq 0\\ \pi-\arcsin\left(\frac{X}{\sqrt{X^2+Y^2}}\right) & ,\quad \text{for} \quad X<0\end{cases} \qquad (20)
$$